

# ON SOME INTEGRAL INEQUALITIES FOR s-LOGARITHMICALLY CONVEX FUNCTIONS AND THEIR APPLICATIONS

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ABSTRACT. In this paper, we describe  $s$ -logarithmically convex functions in the first and second sense which are connected with the ordinary logarithmic convex and  $s$ -convex in the first and second sense. Afterwards, some new inequalities related to above new definitions are given.

## 1. INTRODUCTION

In this section we will present definitions and some results used in this paper. In what follows,  $I$  will be used to denote an interval of real numbers.

**Definition 1.** Let  $I$  be an interval in  $\mathbb{R}$ . Then  $f : I \rightarrow \mathbb{R}$ ,  $\emptyset \neq I \subseteq \mathbb{R}$  is said to be convex if

$$(1.1) \quad f(tx + (1-t)y) \leq tf(x) + (1-t)f(y).$$

for all  $x, y \in I$  and  $t \in [0, 1]$ .

The following concept was introduced by Ozlicz in the paper [2] and was used in the theory of Ozlicz spaces:

**Definition 2.** [2] Let  $0 < s \leq 1$ . A function  $f : \mathbb{R}_+ \rightarrow \mathbb{R}$  where  $\mathbb{R}_+ := [0, \infty)$ , is said to be  $s$ -convex in the first sense if

$$(1.2) \quad f(\alpha u + \beta v) \leq \alpha^s f(u) + \beta^s f(v)$$

for all  $u, v \in \mathbb{R}_+$  and  $\alpha, \beta \geq 0$  with  $\alpha^s + \beta^s = 1$ . It is denoted this by  $f \in K_s^1$ .

In the paper [4], H. Hudzik and L. Maligranda considered, among others, the following class of functions:

**Definition 3.** [4] A function  $f : \mathbb{R}_+ \rightarrow \mathbb{R}$  is said to be  $s$ -convex in the second sense if

$$(1.3) \quad f(\alpha u + \beta v) \leq \alpha^s f(u) + \beta^s f(v)$$

for all  $u, v \geq 0$  and  $\alpha, \beta \geq 0$  with  $\alpha + \beta = 1$  and  $s$  fixed in  $(0, 1]$ . They denoted this by  $f \in K_s^2$ .

It can be easily checked for  $s = 1$ ,  $s$ -convexity reduces to the ordinary convexity of functions defined on  $[0, \infty)$ .

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**Definition 4.** [13] A function  $f : I \rightarrow [0, \infty)$  is said to be *log-convex* or *multiplicatively convex* if  $\log f$  is convex, or equivalently, if for all  $x, y \in I$  and  $t \in [0, 1]$ , one has the inequality:

$$(1.4) \quad f(tx + (1-t)y) \leq [f(x)]^t [f(y)]^{1-t}.$$

We note that if  $f$  and  $g$  are convex functions and  $g$  is monotonic nondecreasing, then  $g \circ f$  is convex. Moreover, since  $f = \exp(\log f)$ , it follows that a log-convex function is convex, but the converse is not true [1.3, p. 7]. This fact is obvious from (1.4) as by the arithmetic-geometric mean inequality, we have

$$(1.5) \quad [f(x)]^t [f(y)]^{1-t} \leq tf(x) + (1-t)f(y)$$

for all  $x, y \in I$  and  $t \in [0, 1]$ .

If the above inequality (1.4) is reversed, then  $f$  is called *logarithmically concave*, or simply *log-concave*. Apparently, it would seem that *log-concave* (*log-convex*) functions would be unremarkable because they are simply related to concave (convex) functions. But they have some surprising properties. It is well known that the product of *log-concave* (*log-convex*) functions is also *log-concave* (*log-convex*). Moreover, the sum of *log-convex* functions is also *log-convex*, and a convergent sequence of *log-convex* (*log-concave*) functions has a *log-convex* (*log-concave*) limit function provided the limit is positive. However, the sum of *log-concave* functions is not necessarily *log-concave*. Due to their interesting properties, the *log-convex* (*log-concave*) functions frequently appear in many problems of classical analysis and probability theory.

The next inequality is well known in the literature as the Hermite-Hadamard inequality for convex functions

$$(1.6) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2}$$

where  $f : I \rightarrow \mathbb{R}$  is a convex function on the interval  $I$  of real numbers  $a, b \in I$  with  $a < b$ .

For some recent results related to this classic result, see the papers [4]-[12] and the books [13]-[16] where further references are given.

In [7], S.S. Dragomir and B. Mond proved that the following inequalities of Hermite-Hadamard type hold for *log-convex* functions:

**Theorem 1.** Let  $f : I \rightarrow [0, \infty)$  be a *log-convex* mapping on  $I$  and  $a, b \in I$  with  $a < b$ . Then one has the inequality:

$$(1.7) \quad f(A(a, b)) \leq \frac{1}{b-a} \int_a^b G(f(x), f(a+b-x)) dx \leq G(f(a), f(b)).$$

**Theorem 2.** Let  $f : I \rightarrow (0, \infty)$  be a *log-convex* mapping on  $I$  and  $a, b \in I$  with  $a < b$ . Then one has the inequality:

$$(1.8) \quad \begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \exp\left[\frac{1}{b-a} \int_a^b \ln[f(x)] dx\right] \\ &\leq \frac{1}{b-a} \int_a^b G(f(x), f(a+b-x)) dx \leq \frac{1}{b-a} \int_a^b f(x) dx \\ &\leq L(f(a), f(b)) \leq \frac{f(a) + f(b)}{2} \end{aligned}$$

where  $G(p, q) := \sqrt{pq}$  is the geometric mean and  $L(p, q) := \frac{p-q}{\ln p - \ln q}$  ( $p \neq q$ ) is the logarithmic mean of the strictly positive real numbers  $p, q$ , i.e.,

$$L(p, q) = \frac{p-q}{\ln p - \ln q} \text{ if } p \neq q \text{ and } L(p, p) = p.$$

In [12], B.G. Pachpatte proved that the inequalities hold for two *log*-convex functions:

$$(1.9) \quad \frac{4}{b-a} \int_a^b f(x) g(x) dx \leq [f(a) + f(b)] L(f(a), f(b)) \\ + [g(a) + g(b)] L(g(a), g(b))$$

Recently, In [3], the concept of geometrically and  $m$ - and  $(\alpha, m)$ -logarithmically convex functions was introduced as follows.

**Definition 5.** A function  $f : [0, b] \rightarrow (0, \infty)$  is said to be  $m$ -logarithmically convex if the inequality

$$(1.10) \quad f(tx + m(1-t)y) \leq [f(x)]^t [f(y)]^{m(1-t)}$$

holds for all  $x, y \in [0, b]$ ,  $m \in (0, 1]$ , and  $t \in [0, 1]$ .

Obviously, if putting  $m = 1$  in Definition 5, then  $f$  is just the ordinary logarithmically convex function on  $[0, b]$ .

**Definition 6.** A function  $f : [0, b] \rightarrow (0, \infty)$  is said to be  $(\alpha, m)$ -logarithmically convex if

$$(1.11) \quad f(tx + m(1-t)y) \leq [f(x)]^{t^\alpha} [f(y)]^{m(1-t^\alpha)}$$

holds for all  $x, y \in [0, b]$ ,  $(\alpha, m) \in (0, 1] \times (0, 1]$ , and  $t \in [0, 1]$ .

Clearly, when taking  $\alpha = 1$  in Definition 6, then  $f$  becomes the standard  $m$ -logarithmically convex function on  $[0, b]$ .

The main purpose of this paper is to introduce the concepts of  $s$ -logarithmically convex in the first and second sense and to establish some inequalities of Hadamard type for  $s$ -logarithmically convex functions.

## 2. DEFINITIONS OF $s$ -LOGARITHMICALLY CONVEX FUNCTIONS

Now it is time to introduce two new classes of functions which will be called  $s$ -logarithmically convex in the first and second sense.

**Definition 7.** A function  $f : I \subset \mathbb{R}_0 \rightarrow \mathbb{R}_+$  is said to be  $s$ -logarithmically convex in the first sense if

$$(2.1) \quad f(\alpha x + \beta y) \leq [f(x)]^{\alpha^s} [f(y)]^{\beta^s}$$

for some  $s \in (0, 1]$ , where  $x, y \in I$  and  $\alpha^s + \beta^s = 1$ .

**Definition 8.** A function  $f : I \subset \mathbb{R}_0 \rightarrow \mathbb{R}_+$  is said to be  $s$ -logarithmically convex in the second sense if

$$(2.2) \quad f(tx + (1-t)y) \leq [f(x)]^{t^s} [f(y)]^{(1-t)^s}$$

for some  $s \in (0, 1]$ , where  $x, y \in I$  and  $t \in [0, 1]$ .

Clearly, when taking  $s = 1$  in Definition 2.1 or Definition 2.2, then  $f$  becomes the standard logarithmically convex function on  $I$ . If the above inequalities (2.1) and (2.2) are reversed, then  $f$  is called  $s$ -logarithmically concave in the first and second sense, respectively.

### 3. ON SOME INEQUALITIES FOR $s$ -LOGARITHMICALLY CONVEX FUNCTION IN THE FIRST SENSE

**Theorem 3.** *Let  $f : I \subset \mathbb{R}_0 \rightarrow \mathbb{R}_+$  be a  $s$ -logarithmically convex mapping in the first sense and monotonic nondecreasing on  $I$  with  $s \in (0, 1]$ . If  $a, b \in I$  with  $a < b$ , then the following inequality holds:*

$$(3.1) \quad f\left(\frac{a+b}{2^{\frac{1}{s}}}\right) \leq \frac{1}{b-a} \int_a^b G(f(x), f(a+b-x)) dx$$

where  $G(p, q) := \sqrt[p]{pq}$  is geometric mean of the strictly positive real numbers  $p, q$ .

*Proof.* If we choose in the definition of  $s$ -logarithmically convex mapping in the first sense  $\alpha = \frac{1}{2^{\frac{1}{s}}}$ ,  $\beta = \frac{1}{2^{\frac{1}{s}}}$ , we have that  $\alpha^s + \beta^s = 1$  and then for all  $x, y \in I$

$$f\left(\frac{x+y}{2^{\frac{1}{s}}}\right) \leq G(f(x), f(y)).$$

If we choose  $x = ta + (1-t)b$ ,  $y = tb + (1-t)a$ ,  $t \in [0, 1]$ , we obtain

$$f\left(\frac{a+b}{2^{\frac{1}{s}}}\right) \leq G(f(ta + (1-t)b), f(tb + (1-t)a)).$$

By integrating over  $t$  on  $[0, 1]$  in the above inequality. The proof is completed.  $\square$

**Theorem 4.** *Let  $f : I \subset \mathbb{R}_0 \rightarrow \mathbb{R}_+$  be a  $s$ -logarithmically convex mapping in the first sense and monotonic nondecreasing on  $I$  with  $s \in (0, 1]$ . If  $a, b \in I$  with  $a < b$ , and  $\frac{1}{p} + \frac{1}{q} = 1$  with  $p < 0$  or  $q < 0$ , then the following inequality holds:*

$$(3.2) \quad \left( \int_0^1 \left[ f\left(ta + (1-t^s)^{\frac{1}{s}}b\right) \right]^p dt \right)^{\frac{1}{p}} \left( \int_0^1 \left[ f\left(ta + (1-t^s)^{\frac{1}{s}}b\right) (1-t^s)^{\frac{1}{s}-1} t^{s-1} \right]^q dt \right)^{\frac{1}{q}} \leq [f(a)][f(b)]$$

*Proof.* If we choose in the Definition 7  $\alpha = t$ ,  $\beta = (1-t^s)^{\frac{1}{s}}$ ,  $t \in [0, 1]$ , we have that  $\alpha^s + \beta^s = 1$  for all  $t \in [0, 1]$  and

$$(3.3) \quad f\left(ta + (1-t^s)^{\frac{1}{s}}b\right) \leq [f(a)]^{t^s} [f(b)]^{(1-t^s)}$$

for all  $t \in [0, 1]$ , and similarly

$$(3.4) \quad f\left((1-t^s)^{\frac{1}{s}}a + tb\right) \leq [f(a)]^{(1-t^s)} [f(b)]^{t^s}$$

for all  $t \in [0, 1]$ .

If we multiply the above two inequalities, we have that

$$f\left(ta + (1-t^s)^{\frac{1}{s}}b\right) f\left((1-t^s)^{\frac{1}{s}}a + tb\right) \leq [f(a)][f(b)]$$

for all  $t \in [0, 1]$ . If we integrate this inequality over  $t$  on  $[0, 1]$ , we get that

$$\int_0^1 f\left(ta + (1-t^s)^{\frac{1}{s}}b\right) f\left((1-t^s)^{\frac{1}{s}}a + tb\right) dt \leq [f(a)][f(b)].$$

Using Hölder inequality, we have

$$(3.5) \leq \left( \int_0^1 \left[ f \left( ta + (1-t^s)^{\frac{1}{s}} b \right) \right]^p dt \right)^{\frac{1}{p}} \left( \int_0^1 \left[ f \left( (1-t^s)^{\frac{1}{s}} a + tb \right) \right]^q dt \right)^{\frac{1}{q}}$$

Let us denote  $u = (1-t^s)^{\frac{1}{s}}$ ,  $t \in [0, 1]$ . Then  $t = (1-u^s)^{\frac{1}{s}}$  and  $dt = -(1-u^s)^{\frac{1}{s}-1} u^{s-1} du$ ,  $u \in (0, 1]$  and then we have the change of variable

$$\begin{aligned} & \int_0^1 f \left( (1-t^s)^{\frac{1}{s}} a + tb \right) dt \\ &= - \int_1^0 f \left( ua + (1-u^s)^{\frac{1}{s}} b \right) (1-u^s)^{\frac{1}{s}-1} u^{s-1} du \\ &= \int_0^1 f \left( ta + (1-t^s)^{\frac{1}{s}} b \right) (1-t^s)^{\frac{1}{s}-1} t^{s-1} dt. \end{aligned}$$

Using the inequality (3.5), we get that

$$\begin{aligned} & \left( \int_0^1 \left[ f \left( ta + (1-t^s)^{\frac{1}{s}} b \right) \right]^p dt \right)^{\frac{1}{p}} \left( \int_0^1 \left[ f \left( ta + (1-t^s)^{\frac{1}{s}} b \right) (1-t^s)^{\frac{1}{s}-1} t^{s-1} \right]^q dt \right)^{\frac{1}{q}} \\ & \leq [f(a)] [f(b)] \end{aligned}$$

and the proof is completed.  $\square$

#### 4. ON SOME INEQUALITIES FOR $s$ -LOGARITHMICALLY CONVEX FUNCTION IN THE SECOND SENSE

**Theorem 5.** Let  $f : I \subset \mathbb{R}_0 \rightarrow \mathbb{R}_+$  be a  $s$ -logarithmically convex mapping in the second sense on  $I$  with  $s \in (0, 1]$ . If  $a, b \in I$  with  $a < b$ , then the following inequality holds:

$$(4.1) \quad \frac{1}{b-a} \int_a^b f(x) dx \leq \int_0^1 [f(a)]^{t^s} [f(b)]^{(1-t)^s} dt.$$

for all  $t \in [0, 1]$ .

*Proof.* Since  $f$  is  $s$ -logarithmically convex mapping in the second sense, we have, for all  $t \in [0, 1]$

$$f(ta + (1-t)b) \leq [f(a)]^{t^s} [f(b)]^{(1-t)^s}$$

Integrating this inequality over  $t$  on  $[0, 1]$ , we get

$$\int_0^1 f(ta + (1-t)b) dt \leq \int_0^1 [f(a)]^{t^s} [f(b)]^{(1-t)^s} dt.$$

As the change of variable  $x = ta + (1-t)b$  gives us that

$$\int_0^1 f(ta + (1-t)b) dt = \frac{1}{b-a} \int_a^b f(x) dx,$$

The proof is completed.  $\square$

**Theorem 6.** *Under the assumptions of Theorem 5, the following inequality holds:*

$$(4.2) \quad \frac{1}{b-a} \int_a^b f(x) dx \leq K(s, k(\mu))$$

where

$$\mu(u, v) = [f(a)]^u [f(b)]^{-v}, u, v > 0,$$

$$k(\mu) = \begin{cases} 1, & \mu = 1, \\ \frac{\mu-1}{\ln \mu}, & \mu \neq 1, \end{cases}$$

and

$$K(s, k(\mu)) = [f(b)]^s k(\mu(s, s)), \quad f(a), f(b) \leq 1.$$

*Proof.* Since  $f$  is  $s$ -logarithmically convex mapping in the second sense, we have, for all  $t \in [0, 1]$

$$f(ta + (1-t)b) \leq [f(a)]^{t^s} [f(b)]^{(1-t)^s}$$

Integrating this inequality over  $t$  on  $[0, 1]$ , we get

$$\int_0^1 f(ta + (1-t)b) dt \leq \int_0^1 [f(a)]^{t^s} [f(b)]^{(1-t)^s} dt.$$

If  $0 < \rho \leq 1$ ,  $0 < t, s \leq 1$ , then

$$(4.3) \quad \rho^{t^s} \leq \rho^{ts}.$$

When  $f(a), f(b) \leq 1$ , by (4.3), we get that

$$(4.4) \quad \begin{aligned} \int_0^1 [f(a)]^{t^s} [f(b)]^{(1-t)^s} dt &\leq \int_0^1 [f(a)]^{st} [f(b)]^{s(1-t)} dt \\ &= [f(b)]^s \int_0^1 [f(a)]^{st} [f(b)]^{-st} dt \\ &= [f(b)]^s k(\mu(s, s)). \end{aligned}$$

As the change of variable  $x = ta + (1-t)b$  gives us that

$$(4.5) \quad \int_0^1 f(ta + (1-t)b) dt = \frac{1}{b-a} \int_a^b f(x) dx,$$

from (4.4) to (4.5), (4.2) holds.  $\square$

**Theorem 7.** *Let  $f, g : I \subset \mathbb{R}_0 \rightarrow \mathbb{R}_+$  be a  $s$ -logarithmically convex mappings in the second sense on  $I$  with  $s \in (0, 1]$ . If  $a, b \in I$  with  $a < b$ , then the following inequality holds:*

$$(4.6) \quad \frac{1}{b-a} \int_a^b f(x) g(x) dx \leq K(s, k(\eta))$$

where

$$\eta(u, v) = [f(a)g(a)]^u [f(b)g(b)]^{-v}, u, v > 0,$$

$$k(\eta) = \begin{cases} 1, & \eta = 1, \\ \frac{\eta-1}{\ln \eta}, & \eta \neq 1, \end{cases}$$

and

$$K(s, k(\eta)) = [f(b)g(b)]^s k(\eta(s, s)), \quad f(a)g(a), f(b)g(b) \leq 1.$$

*Proof.* Since  $f, g$  are  $s$ -logarithmically convex mappings in the second sense, we have, for all  $t \in [0, 1]$

$$f(ta + (1-t)b) \leq [f(a)]^{t^s} [f(b)]^{(1-t)^s} \text{ and } g(ta + (1-t)b) \leq [g(a)]^{t^s} [g(b)]^{(1-t)^s}$$

from which it follows that

$$\begin{aligned} (4.7) \quad \int_a^b f(x) g(x) dx &= (b-a) \int_0^1 f(ta + (1-t)b) g(ta + (1-t)b) dt \\ &\leq (b-a) \int_0^1 [f(a) g(a)]^{t^s} [f(b) g(b)]^{(1-t)^s} dt \end{aligned}$$

When  $f g(a), f g(b) \leq 1$ , by (4.3), we get that

$$\begin{aligned} (4.8) \quad \int_0^1 [f(a) g(a)]^{t^s} [f(b) g(b)]^{(1-t)^s} dt &\leq \int_0^1 [f(a) g(a)]^{st} [f(b) g(b)]^{s(1-t)} dt \\ &= [f(b) g(b)]^s k(\eta(s, s)). \end{aligned}$$

from (4.7) to (4.8), (4.6) holds.  $\square$

**Corollary 1.** Let  $f, g : I \subset \mathbb{R}_0 \rightarrow \mathbb{R}_+$  be a log-convex mappings on  $I$ . If  $a, b \in I$  with  $a < b$ , then the following inequality holds:

$$\frac{1}{b-a} \int_a^b f(x) g(x) dx \leq L(f(a) g(a), f(b) g(b))$$

where  $L(p, q) := \frac{p-q}{\ln p - \ln q}$  ( $p \neq q$ ) is the logarithmic mean of the positive real numbers.

*Proof.* We take  $s = 1$  in (4.6), we get the required result.  $\square$

**Theorem 8.** Let  $f, g, a, b$  be as in Theorem 7 and  $\alpha, \beta > 0$  with  $\alpha + \beta = 1$ . Then the following inequality holds:

$$(4.9) \quad \frac{1}{b-a} \int_a^b f(x) g(x) dx \leq K(s, \alpha, \beta; k(\omega), k(\ell))$$

where

$$\omega(u, v) = [f(a)]^u [f(b)]^{-v}, \text{ and } \ell(u, v) = [g(a)]^u [g(b)]^{-v}, \quad u, v > 0,$$

$$k(\omega) = \begin{cases} 1, & \omega = 1, \\ \frac{\omega-1}{\ln \omega}, & \omega \neq 1, \end{cases}, \quad k(\ell) = \begin{cases} 1, & \ell = 1, \\ \frac{\ell-1}{\ln \ell}, & \ell \neq 1, \end{cases}$$

and

$$\begin{aligned} &K(s, \alpha, \beta; k(\omega), k(\ell)) \\ &= \alpha [f(b)]^{\frac{s}{\alpha}} k\left(\omega\left(\frac{s}{\alpha}, \frac{s}{\alpha}\right)\right) + \beta [g(b)]^{\frac{s}{\beta}} k\left(\ell\left(\frac{s}{\beta}, \frac{s}{\beta}\right)\right), \quad f(a), g(a), f(b), g(b) \leq 1. \end{aligned}$$

*Proof.* Since  $f, g$  are  $s$ -logarithmically convex mappings in the second sense, we have, for all  $t \in [0, 1]$

$$(4.10) \quad f(ta + (1-t)b) \leq [f(a)]^{t^s} [f(b)]^{(1-t)^s} \quad \text{and} \quad g(ta + (1-t)b) \leq [g(a)]^{t^s} [g(b)]^{(1-t)^s}$$

from which it follows that

$$(4.11) \quad \int_a^b f(x) g(x) dx = (b-a) \int_0^1 f(ta + (1-t)b) g(ta + (1-t)b) dt$$

Using the well known inequality  $mn \leq \alpha m^{\frac{1}{\alpha}} + \beta n^{\frac{1}{\beta}}$ , (4.10), on the right side of (4.11), we have

$$\begin{aligned} \frac{1}{b-a} \int_a^b f(x) g(x) dx &\leq \int_0^1 \left\{ \alpha [f(ta + (1-t)b)]^{\frac{1}{\alpha}} + \beta [g(ta + (1-t)b)]^{\frac{1}{\beta}} \right\} dt \\ &\leq \int_0^1 \left\{ \alpha \left[ [f(a)]^{t^s} [f(b)]^{(1-t)^s} \right]^{\frac{1}{\alpha}} + \beta \left[ [g(a)]^{t^s} [g(b)]^{(1-t)^s} \right]^{\frac{1}{\beta}} \right\} dt \\ &= \alpha \int_0^1 [f(a)]^{\frac{t^s}{\alpha}} [f(b)]^{\frac{(1-t)^s}{\alpha}} dt + \beta \int_0^1 [g(a)]^{\frac{t^s}{\beta}} [g(b)]^{\frac{(1-t)^s}{\beta}} dt \end{aligned}$$

When  $f(a), g(a), f(b), g(b) \leq 1$ , by (4.3), we get that

$$\begin{aligned} \alpha \int_0^1 [f(a)]^{\frac{t^s}{\alpha}} [f(b)]^{\frac{(1-t)^s}{\alpha}} dt &\leq \alpha \int_0^1 [f(a)]^{\frac{st}{\alpha}} [f(b)]^{\frac{s(1-t)}{\alpha}} dt \\ &= \alpha [f(b)]^{\frac{s}{\alpha}} k\left(\omega\left(\frac{s}{\alpha}, \frac{s}{\alpha}\right)\right). \\ \beta \int_0^1 [g(a)]^{\frac{t^s}{\beta}} [g(b)]^{\frac{(1-t)^s}{\beta}} dt &\leq \beta \int_0^1 [g(a)]^{\frac{st}{\beta}} [g(b)]^{\frac{s(1-t)}{\beta}} dt \\ (4.12) \quad &= \beta [g(b)]^{\frac{s}{\beta}} k\left(\ell\left(\frac{s}{\beta}, \frac{s}{\beta}\right)\right). \end{aligned}$$

From (4.11) to (4.12), (4.9) holds.  $\square$

**Corollary 2.** Let  $f, g : I \subset \mathbb{R}_0 \rightarrow \mathbb{R}_+$  be a log-convex mappings on  $I$ . If  $a, b \in I$  with  $a < b$ , then the following inequality holds:

$$\frac{1}{b-a} \int_a^b f(x) g(x) dx \leq \alpha \times L\left([f(a)]^{\frac{1}{\alpha}}, [f(b)]^{\frac{1}{\alpha}}\right) + \beta \times L\left([g(a)]^{\frac{1}{\beta}}, [g(b)]^{\frac{1}{\beta}}\right)$$

where  $L(p, q) := \frac{p-q}{\ln p - \ln q}$  ( $p \neq q$ ) is the logarithmic mean of the positive real numbers.

*Proof.* We take  $s = 1$  in (4.9), we get the required result.  $\square$

**Theorem 9.** Let  $f, g, a, b$  be as in Theorem 7 and  $p, q > 1$  with  $\frac{1}{p} + \frac{1}{q} = 1$ . Then the following inequality holds:

$$(4.13) \quad \frac{1}{b-a} \int_a^b f(x) g(x) dx \leq K(s, p, q; k(\omega), k(\ell))$$



where  $\omega(u, v)$ ,  $\ell(u, v)$ ,  $k(\omega)$ ,  $k(\ell)$  is defined as above and

$$\begin{aligned} & K(s, p, q; k(\omega), k(\ell)) \\ = & [f(b)g(b)]^s (k(\omega(sp, sp)))^{\frac{1}{p}} (k(\ell(sq, sq)))^{\frac{1}{q}}, \quad f(a), g(a), f(b), g(b) \leq 1. \end{aligned}$$

*Proof.* Since  $f$  and  $g$  are positive funtions and using the well known Hölder inequality on the right side of (4.11), we have

$$\begin{aligned} (4.14) \quad & \frac{1}{b-a} \int_a^b f(x)g(x)dx \\ & \leq \left( \int_0^1 [f(ta + (1-t)b)]^p dt \right)^{\frac{1}{p}} \left( \int_0^1 [g(ta + (1-t)b)]^q dt \right)^{\frac{1}{q}} \\ & \leq \left( \int_0^1 [f(a)]^{pt^s} [f(b)]^{p(1-t)^s} dt \right)^{\frac{1}{p}} \left( \int_0^1 [g(a)]^{qt^s} [g(b)]^{q(1-t)^s} dt \right)^{\frac{1}{q}} \end{aligned}$$

When  $0 < f(a), g(a), f(b), g(b) \leq 1$ , by (4.3), we get that

$$\begin{aligned} \int_0^1 [f(a)]^{pt^s} [f(b)]^{p(1-t)^s} dt & \leq \int_0^1 [f(a)]^{spt} [f(b)]^{sp(1-t)} dt \\ & = [f(b)]^{sp} k(\omega(sp, sp)). \\ \int_0^1 [g(a)]^{qt^s} [g(b)]^{q(1-t)^s} dt & \leq \int_0^1 [g(a)]^{sqt} [g(b)]^{sq(1-t)} dt \\ (4.15) \quad & = [g(b)]^{sq} k(\ell(sq, sq)). \end{aligned}$$

Therefore,

$$\begin{aligned} & \frac{1}{b-a} \int_a^b f(x)g(x)dx \\ & \leq [f(b)g(b)]^s (k(\omega(sp, sp)))^{\frac{1}{p}} (k(\ell(sq, sq)))^{\frac{1}{q}}. \end{aligned}$$

Thus, Theorem 9 is proved.  $\square$

**Corollary 3.** Let  $f, g : I \subset \mathbb{R}_0 \rightarrow \mathbb{R}_+$  be a log-convex mappings on  $I$ . If  $a, b \in I$  with  $a < b$ , then the following inequality holds:

$$\frac{1}{b-a} \int_a^b f(x)g(x)dx \leq (L([f(a)]^p, [f(b)]^p))^{\frac{1}{p}} (L([g(a)]^q, [g(b)]^q))^{\frac{1}{q}}$$

where  $L(p, q) := \frac{p-q}{\ln p - \ln q}$  ( $p \neq q$ ) is the logarithmic mean of the positive real numbers.

*Proof.* We take  $s = 1$  in (4.13), we get the required result.  $\square$

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